General relativity, Lauricella's hypergeometric function F_D and the theory of braids

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Abstract

The exact (closed form) solutions of the equations of motion in the theory of general relativity that describe motion of test particle and photon in Kerr and Kerr-(anti) de Sitter spacetimes all involve the multivariable hypergeometric function of Lauricella F_D: Kraniotis [Class. Quantum Grav. **21** 2004, 4743; Class. Quantum Grav. **22** 2005, 4391; Class. Quantum Grav. **24** 2007, 1775]. The domain of variables $\mathcal{D}_n$ of the corresponding function depends on the first integrals of motion associated with the isometries of the Kerr-(anti) de Sitter metric and Carter's constant Q as well as on the cosmological constant Λ and the Kerr (rotation) parameter. In this work we discuss the topological properties of the domain $\mathcal{D}_n$ and in particular its fundamental connection with the theory of braids. An intrinsic relationship of general relativity with the pure braids is established.

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1 Introduction

1.1 Motivation

The exact (closed form) solutions of the equations of motion of the theory of general relativity (GTR) that describe orbits of test particle and photon in Kerr and Kerr-(anti) de Sitter spacetimes have yielded the following result [1],[2], [3]:

All the physical amplitudes (measurable quantities) related to test particle orbits such as periapsis and gravitomagnetic (Lense-Thirring) precessions, orbital periods as well as the bending of light by a rotating central mass (rotating black hole or rotating star), the gravitomagnetic precessions and orbital periods of spherical photon orbits in Kerr spacetime with a cosmological constant have been elegantly expressed in terms of Lauricella's multivariable hypergeometric function $F_D(\alpha, \beta_1, \beta_2, \cdots, \beta_m, \gamma; z_1, z_2, \cdots, z_m)$. The domain of variables (moduli) of Lauricella's function F_D [4, 5]

$$\mathcal{D}_n = (z_1, z_2, \cdots, z_n); z_i \neq 0, 1, (1 \leq i \leq n), z_j \neq z_k (1 \leq j < k \leq n) \tag{1}$$

is related in the theory of General Relativity through the exact solutions of the geodesics system in Kerr-(anti) de Sitter spacetime to the first integrals of motion, as well as to the cosmological constant Λ and the rotation (Kerr) parameter [1],[2],[3]. The generalised hypergeometric function of Appell-Lauricella F_D is a very important function in Mathematical Analysis and as we shall see in this work it possesses very interesting topological properties. This then can lead to a fundamental relationship of General Relativity with topology. The establishment of such a relationship is the main theme and objective of this work.

Indeed in pure Mathematics the domain of variables of F_D has been studied [6]; a main result of this investigation was the very interesting topological properties of the domain $\mathcal{D}_n$. Essentially, the fundamental group of the domain under discussion, $\pi_1(\mathcal{D}_n, a)$, crudely speaking is the pure (or coloured) symmetry braid group [6]. Thus the results of [1],[2],[3] combined with the previous result constitute a profound and intrinsic relation of the theory of General Relativity with the field of algebraic topology and in particular with the theory of braids and links [1]. The *first*, as a matter of fact, *direct connection* of a theory of Physics with the theory of braids [2].

The material of this paper is organized as follows: In section 2 we present some basic results of braid theory (which are useful in understanding the main result); Namely the presentation theory of the braid and pure braid group as well as their representation theory through the Burau and Gassner matrices respectively (subsections 2.1.1 and 2.2). Having discussed the closure operation and the Markov moves in section 2.1, the topology invariants for knots and links

[1] As we shall see in the main body of the paper the closure of braids are knots and links. In particular, the closure of a coloured braid is a link.

[2] At this point we must mention that nice accounts of previous efforts and results connecting certain models in physics with topological invariants can be found in [7, 8].

associated with the discovery of Jones, Bracket and HOMFLY polynomials are briefly discussed and reviewed in 3. In section 4 we discuss the presentation of $\pi_1(\mathcal{D}_n, a)$ and the hypergeometric representation of the pure or coloured group that it defines, an approach developed in [6]. Combining with the results in [1],[2],[3] the establishment of the fundamental connection of the theory of General Relativity with the coloured braids via the hypergeometric function of Lauricella F_D is then achieved. Finally, section 5 is used for our conclusions.

2 The theory of braids and their symmetry group

Braids (Zöpfe) are very beautiful and profound mathematical entities. They have been constructed by the German mathematician Emil Artin [9],[10] first as an application to textile industry evolving into a central theme of topology where they currently serve as the fundamental theory of knots and links.

In the space $\mathbb{R}^3$, consider the points $A_i = (i, 0, 0)$ and $B_i = (i, 0, 1)$, where $i = 1, 2, \ldots, n$. A polygonal line joining one of the points A_i with one of the points B_j is called ascending if in the motion of a point from A_i to B_j along the line its z-coordinate increases monotonically. A braid in n strands (or strings) is defined as a set of pairwise nonintersecting ascending polygonal lines (the strands) joining the points $A_1, \ldots, A_n$ to the points $B_1, \ldots, B_n$ (in any order). One can also consider braids whose strands are ascending smooth lines (rather than polygonal ones); then it is natural to define equivalence as isotopy, i.e., as a smooth deformation in the class of braids. Examples of braids are given in fig.1.

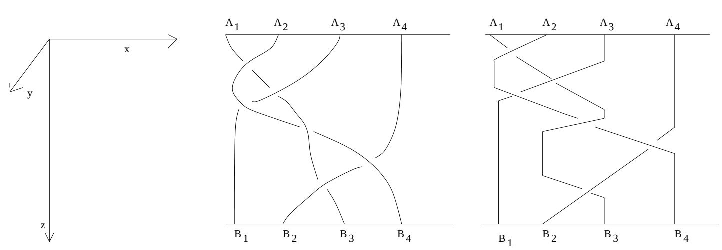


Figure 1: Examples of braids.

Braids form a group and we now discuss its properties.

We denote by σ_i the braid which joins i to $i+1$ by a path passing under the path joining $i+1$ to i (see Figure 2).

The braid group B_n is generated by the elements $\sigma_1, \sigma_2, \cdots \sigma_{n-1}$ with the following presentation [9]:

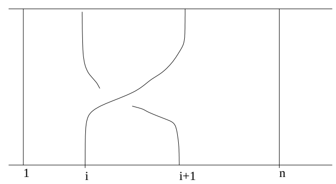


Figure 2: The generator σ_i.

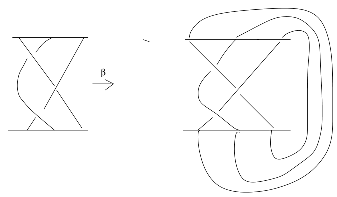


Figure 3: Closure of a braid

$$
\begin{aligned}
\text{Generators :} \quad & \sigma_1, \sigma_2, \cdots \sigma_{n-1} \\
\text{Relations :} \quad & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (i = 1, \cdots, n-2) \\
& \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1
\end{aligned} \tag{2}
$$

The second relation is sometimes called *far commutativity*, because it says the generators commute pairwise when they are sufficiently far from each other, i.e., when their indices differ by two or more.

2.1 Relationship between braids and links and the closure operation

In this section we discuss the relationship betweeen braids and links arising from the closure operation, which assigns a knot, link to each braid in a natural way.

There is a *canonical epimorphism* $\varphi : B_n \to S_n$ of the braid group onto the permutation group. In terms of relations, the group S_n is obtained from B_n by adding the relation $\sigma_i^2 = 1$. A link or knot $\beta(b)$ is obtained by closing the braid b, i.e., tying the top end of each string (strand) to the same position at the bottom of the braid as shown in fig.3. The closure $\beta(b)$ of the braid b is a knot if the permutation $\varphi(b)$ associated to the braid generates the cyclic subgroup of order n, $\mathbb{Z}/n\mathbb{Z}$, in the permutation group S_n [8].

Next we discuss an important theorem due to A. Markov which answers the question of when different braids can have isotopic closures, i.e. represent the same knot, link. The *first Markov move* replaces $b \in B_n$ by aba^{-1} for $a \in B_n$.

The *second Markov move* is the replacement $b \leftrightarrow b\sigma_n^{\pm 1}$ for $b \in B_n$ (note that $\sigma_n \notin B_n$, so the notation $b\sigma_n$ makes sense algebraically only if we identify b with its image under the natural inclusion $B_n \hookrightarrow B_{n+1}$) Then the theorem asserts that the closures of two braids are isotopic if and only if one braid can be taken to another by a finite sequence of Markov moves. A proof of this theorem can be found in the book of J. S. Birman [11].

Despite the difficulty of applying Markov's theorem for studying knots via braids, braids had first suggested themselves as a useful tool for investigating further their relationship with links via the closure operation, after the discovery by Werner Burau of a matrix representation of B_n. This representation is the subject of the following section

2.1.1 The Burau representation

If x is a non-zero complex number let M_i, for $1 \leq i \leq n-1$, be the $n \times n$ matrix

$$\begin{array}{c} \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1-x & x & & & \\ & & 1 & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} = M_i$$

where $1-x$ is the $i-i$ entry. One may easily check that $M_iM_{i+1}M_i = M_{i+1}M_iM_{i+1}$ and $M_iM_j = M_jM_i$ if $|i-j| \geq 2$. Thus sending σ_i to M_i defines the (non-reduced) Burau representation of B_n [12]. Burau recognized that his representation was related to closed braids. More specifically if $\alpha \in B_n$ and ψ is the reduced Burau representation then $\det(1-\psi(\alpha))$ is $(1+x+\ldots+x^{n-1})$ times the Alexander polynomial of the link $\hat{\alpha}$ [13].

2.2 The coloured braid group symmetry

The kerner of the epimorphism φ defines the *coloured braid symmetry group* or *pure braid group* P_n [3]

$$P_n = \mathrm{Ker}\varphi \tag{3}$$

We first define generators for P_n.

For any $i<j$, set $A_{ij} = A_{ji} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$.

The generator A_{ij} is depicted in Figure 4.

Artin [10] gives the following presentation for P_n [4]

[3] As a matter of fact, the following short exact sequence is valid: $1 \to P_n \xrightarrow{\rho} B_n \xrightarrow{\varphi} S_n \to 1$.

[4] One way to obtain a presentation for P_n is using the Schreier-Reidemeister method. The group P_n is of index $n!$ in B_n. One may choose as coset representatives for P_n in B_n any set

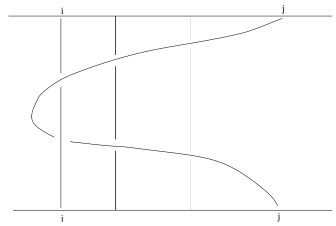


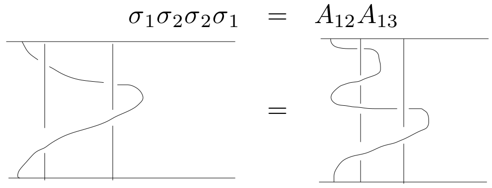
Figure 4: The generator A_{ij} for the pure symmetry braid group.

$$
\begin{array}{l}
\text{Generators} : A_{ij} ; 1 \leq i < j < n \\
\text{Relations} : \\
1) \quad A_{rs}^{\epsilon} A_{ik} A_{rs}^{-\epsilon} = A_{ik} \\
\text{if all indices are different} \\
\text{and if the pairs } r, s \; and \; i, k \text{ do not separate each other} \\
2) \quad A_{rs}^{\epsilon} A_{ir} A_{rs}^{-\epsilon} = A_{is}^{-\epsilon} A_{ir} A_{is}^{\epsilon} \\
3) \quad A_{rs}^{\epsilon} A_{is} A_{rs}^{-\epsilon} = A_{is}^{-\epsilon} A_{ir}^{-\epsilon} A_{is} A_{ir}^{\epsilon} A_{is}^{\epsilon} \\
\text{if finally the subscripts are all different and} \\
\text{the pairs } r, s \text{ and } i, k \text{ separate each other we get} \\
4) \quad A_{rs}^{\epsilon} A_{ik} A_{rs}^{-\epsilon} = A_{is}^{-\epsilon} A_{ir}^{-\epsilon} A_{is}^{\epsilon} A_{ir}^{\epsilon} . A_{ik} . A_{ir}^{-\epsilon} A_{is}^{-\epsilon} A_{ir}^{\epsilon} A_{is}^{\epsilon}
\end{array}
\tag{4}
$$

and $\epsilon = \pm 1$.

Every pure braid can be combed i.e. it can be represented in terms of the generators A_{ij}

$$\sigma_1 \sigma_2 \sigma_2 \sigma_1 \quad = \quad A_{12} A_{13}$$



(5)

The element $(A_{12})(A_{13}A_{23}) \cdots (A_{1n}A_{2n} \cdots A_{n-1n}) \in$ centre of P_n.

Let us give an example for $n = 4$, the symmetry group P_4 has presentation

of $n!$ words in the generators of B_n whose images under φ range over all of S_n. When these coset representatives form a Schreier set (i.e. any initial segment of a coset representative is again a coset representative) one can apply the Schreier-Reidemeister method [14].

$$
\begin{aligned}
&\text{Generators}: A_{12}, A_{23}, A_{34}, A_{13}, A_{24}, A_{14}\\
&\text{Relations}:\\
&A_{12}A_{34} = A_{34}A_{12}\\
&A_{14}A_{23} = A_{23}A_{14}\\
&A_{24}A_{13}A_{24}^{-1} = A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{13}A_{12}^{-1}A_{14}^{-1}A_{12}A_{14}\\
&A_{23}A_{12}A_{23}^{-1} = A_{13}^{-1}A_{12}A_{13}\\
&A_{24}A_{12}A_{24}^{-1} = A_{14}^{-1}A_{12}A_{14}\\
&A_{34}A_{13}A_{34}^{-1} = A_{14}^{-1}A_{13}A_{14}\\
&A_{34}A_{23}A_{34}^{-1} = A_{24}^{-1}A_{23}A_{24}\\
&A_{23}A_{13}A_{23}^{-1} = A_{13}^{-1}A_{12}^{-1}A_{13}A_{12}A_{13}\\
&A_{24}A_{14}A_{24}^{-1} = A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{14}\\
&A_{34}A_{14}A_{34}^{-1} = A_{14}^{-1}A_{13}^{-1}A_{14}A_{13}A_{14}\\
&A_{34}A_{24}A_{34}^{-1} = A_{24}^{-1}A_{23}^{-1}A_{24}A_{23}A_{24}
\end{aligned}
\tag{6}
$$

It can easily be checked that the following matrices constitute a representation of the pure braid group P_4, i.e. they satisfy the relations (6).

$$
A_{12} = \begin{pmatrix} 1-x_1+x_1x_2 & (1-x_1)x_1 & 0 & 0 \\ 1-x_2 & x_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
A_{13} = \begin{pmatrix} 1-x_1+x_1x_3 & 0 & (1-x_1)x_1 & 0 \\ (1-x_2)(1-x_3) & 1 & (-1+x_1)(1-x_2) & 0 \\ 1-x_3 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
A_{14} = \begin{pmatrix} 1-x_1+x_1x_4 & 0 & 0 & (1-x_1)x_1 \\ (1-x_2)(1-x_4) & 1 & 0 & (-1+x_1)(1-x_2) \\ (1-x_3)(1-x_4) & 0 & 1 & (-1+x_1)(1-x_3) \\ 1-x_4 & 0 & 0 & x_1 \end{pmatrix}
$$

$$
A_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+x_2(-1+x_3) & (1-x_2)x_2 & 0 \\ 0 & 1-x_3 & x_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
A_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-x_2+x_2x_4 & 0 & (1-x_2)x_2 \\ 0 & (1-x_3)(1-x_4) & 1 & (-1+x_2)(1-x_3) \\ 0 & 1-x_4 & 0 & x_2 \end{pmatrix}
$$

$$A_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-x_3+x_3x_4 & (1-x_3)x_3 \\ 0 & 0 & 1-x_4 & x_3 \end{pmatrix}$$

This matrix representation is the famous representation discovered by Betty Jane Gassner in 1961 [15], generalizing the Burau representation.

The element $A_{12}A_{13}A_{23}A_{14}A_{24}A_{34}$ is represented by the matrix

$$\begin{pmatrix} 1+x_1(-1+x_2x_3x_4) & -(-1+x_1)x_1 & -(-1+x_1)x_1x_2 & -(-1+x_1)x_1x_2x_3 \\ 1-x_2 & x_1(1+x_2(-1+x_3x_4)) & -x_1(-1+x_2)x_2 & -x_1(-1+x_2)x_2x_3 \\ 1-x_3 & x_1-x_1x_3 & x_1x_2(1+x_3(-1+x_4)) & -x_1x_2(-1+x_3)x_3 \\ 1-x_4 & x_1-x_1x_4 & -x_1x_2(-1+x_4) & x_1x_2x_3 \end{pmatrix}$$

More generally, denoting by $A_{rs}, 1 \leq r < s \leq n$, the generators of P_n, the (unreduced) Gassner representation is the homomorphism $G_n : P_n \to GL_n(\mathbb{Z}[x_1^{\pm}, \cdots, x_n^{\pm}])$ given by the formula [5]

$$G_n(A_{rs}) = \begin{pmatrix} \mathbb{I}_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 1-x_r+x_rx_s & 0 & x_r(1-x_r) & 0 \\ 0 & \vec{u} & \mathbb{I}_{s-r-1} & \vec{v} & 0 \\ 0 & 1-x_s & 0 & x_r & 0 \\ 0 & 0 & 0 & 0 & \mathbb{I}_{n-s} \end{pmatrix}$$

where

$$\vec{u} = ((1-x_{r+1})(1-x_s)\cdots(1-x_{s-1})(1-x_s))^{\top} \tag{7}$$

and

$$\vec{v} = ((1-x_{r+1})(x_r-1)\cdots(1-x_{s-1})(x_r-1))^{\top} \tag{8}$$

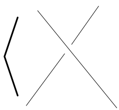
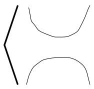
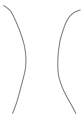
and $\mathbb{I}_k$ denotes the $k \times k$ identity matrix.

3 The Bracket polynomial

Jones introduced his polynomial invariant for tame oriented links via certain representations of the braid group [16] [6], exploiting the similarity of the Ocneanu trace in Hecke algebras with the Markov moves, first pointed out to him by Joan S Birman. There are two ways to introduce the Jones [16] and HOMFLY (or LYMPHTOFU) polynomials [17]. First through braids (each knot and link expressed as a word in the generators of the braid group) [16], and second through the bracket polynomial due to L Kauffman [18]. We briefly discuss both approaches beginning with the definition and properties of the bracket polynomial [18].

[5] The faithfulness of G_n for $n \geq 4$ is an important issue in braid theory.
[6] Different from the Burau representation in section 2.1.1.

To each nonoriented link diagram L a polynomial in the variables a, b, c is assigned, denoted by $< L >$ which satisfies the following defining relations

$\Big\langle$  $\Big\rangle = a \Big\langle$  $\Big\rangle + b \Big\langle$  $\Big\rangle$ (9)

$$< L \sqcup 0 >= c < L > \tag{10}$$

$$< 0 >= 1 \tag{11}$$

Here the little pictures in (9) denote three link diagrams L, L_A, L_B which are identical outside a small disk and are as shown in the picture 5 inside it.

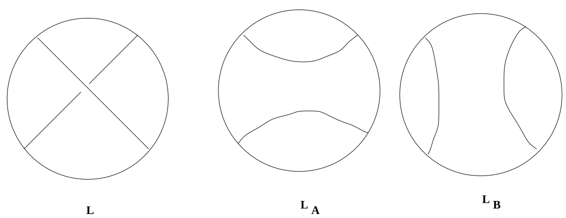


Figure 5: Eliminating a crossing point.

In this notation (9) may be rewritten as $< L >= a < L_A > +b < L_B >$. The arcs inside the small disks of the diagrams L_A and L_B are chosen in the regions A and B defined in Figure 6

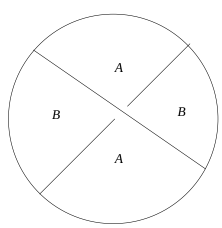
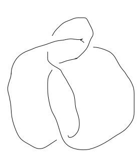


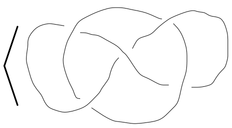
Figure 6: A and B regions near a crossing point.

It turns out that the bracket polynomial is invariant under two of the Reidemeister moves which imposes the constraints $b = a^{-1}, c = -a^2 - b^2$.

For instance for the knot 4_1 the bracket polynomial is calculated as follows

$\Big\langle$  $\Big\rangle = 1 + a^{-8} + a^8 - a^4 - a^{-4}$ (12)

while for the link below we have the result

$\Bigg\langle$  $\Bigg\rangle = -a^{11} + 2a^{7} - a^{3} + 2a^{-1} - a^{-5} + a^{-9} \qquad (13)$

For the Kauffman polynomial one needs to consider *oriented* links, i.e. we assume that each component is supplied with an orientation (shown by arrows in the figures). The *writhe number* is defined as follows

$$\omega(L) := \sum_i \epsilon_i \tag{14}$$

where the sum is taken over all crossing points and the numbers ϵ_i are equal to ± 1 depending on the sign of the ith crossing point, which is defined in figure 7.

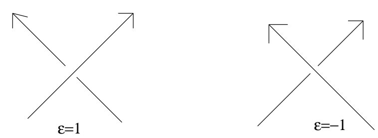
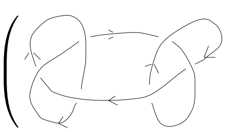
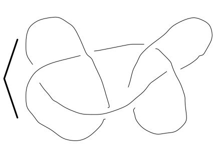


Figure 7: Positive and negative crossing points.

Then the Kauffman polynomial $X(L)$ on any oriented link diagram L is defined by [18]

$$X(L) := (-a)^{-3\omega(L)} < |L| > \tag{15}$$

where the nonoriented diagram $|L|$ is obtained from L by forgetting the orientation of all components. Now for the knot below the Kauffman polynomial is calculated to be

$X\Bigg($  $\Bigg) = \Bigg\langle$  $\Bigg\rangle$

$$= 3 - a^{-4} - a^{4} + a^{-8} + a^{8} - a^{-12} - a^{12} \tag{16}$$

since $\omega(L) = 0$.

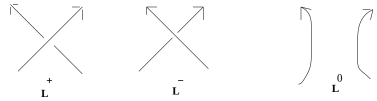
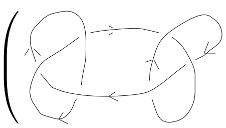


Figure 8: Orientation diagrams in the defining relation for $X(L)$.

Substituting $a = q^{-1/4}$ into $X(L)$ one obrains $V(L)$ the *Jones polynomial* of the oriented link L. For the reef knot in eq.(16) we have

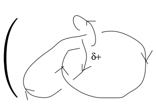
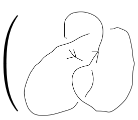
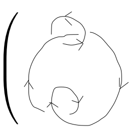
$V\Big($  $\Big) = 3 - q - q^{-1} + q^2 + q^{-2} - q^3 - q^{-3}$ (17)

The Jones polynomial satisfies the following relations [16]

$$\begin{aligned} q^{-1}V(L^+) - qV(L^-) &= (q^{1/2} - q^{-1/2})V(L^0) && (18)\\ V(L \sqcup 0) &= -(q^{-1/2} + q^{1/2})V(L) && (19)\\ V(0) &= 1 && (20) \end{aligned}$$

Equation (18) is known as the *skein relation*, L^+, L^-, L^0 are the three link diagrams exhibited in figure 8 . The condition $L \sqcup 0$ stands for the link L with an added circle that does not intersect L (and has no crossing points with L). The last condition says that the Jones polynomial of the circle is 1.

Using equations (18)-(20) one can calculate in an alternative way the polynomial $V(L)$. Indeed, the skein relation for the figure 8 knot reads

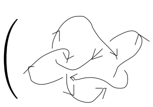
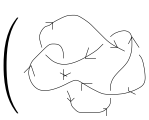
$q^{-1}V\Big($  $\Big) - qV\Big($  $\Big) = (q^{1/2} - q^{-1/2})V\Big($  $\Big)$ (21)

or

$V\Big($  $\Big) = 1 + q^2 + q^{-2} - q - q^{-1}$ (22)

where we used the fact that $V(L)$ for the Hopf link in equation (21) is equal to: $-q^{-1/2} - q^{-5/2}$. Our result using the skein relations agrees with the previous calculation equation (12), where the rules for the bracket polynomial have been applied.

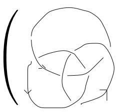
Applying the skein relation for the Stevedore's knot once we obtain

$-qV\Big($  $\Big) + q^{-1}V\Big($  $\Big) = (q^{1/2} - q^{-1/2})V\Big($  $\Big)$ (23)

The right hand side of equation (23) is a Hopf link with Jones polynonial: $-q^{5/2} - q^{1/2}$. Applying the skein relations to the second member of the left hand side of (23) we obtain the unknot and a Hopf link. Eventually we obtain for the Jones polynomial of the Stevedore's (6_1) knot

$V\Big($  $\Big) = 2 - q - 2q^{-1} + q^{-2} + q^2 - q^{-3} + q^{-4}$ (24)

As a final example the Jones polynomial for the link of the Borromean rings is calculated to be:

$V\Big($  $\Big) = -q^3 + 3q^2 - 2q + 4 - 2q^{-1} + 3q^{-2} - q^{-3} \qquad (25)$

3.1 Braid words for links, knots and the HOMFLY polynomial

It is useful to express the links and knots we discussed in the previous section in words in terms of the generators of the braid group. The braid word for the Hopf link, in terms of generators of the standard and pure braid group is: $\sigma_1^2 = A_{12}$ while for the Borromeo rings which appear in Eq. (25) the corresponding word is given by: $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} = A_{12}A_{13}^{-1}A_{12}^{-1}A_{13}$ On the other hand for the reef or square knot in eq.(16) the braid word is $\sigma_1^{-3}\sigma_2^3$, i.e. a braid with 3-strings.

The two-variable HOMFLY polynomial of the oriented link L is defined as follows [17],[19]

$$X_L(x,\lambda) := \left(-\frac{1-\lambda x}{\sqrt{\lambda}(1-x)}\right)^{n-1} (\sqrt{\lambda})^e \mathrm{tr}(\pi(b)) \qquad (26)$$

where $b \in B_n$ is any braid with $\beta(b) = L$, e being the exponent sum of b as a word on the σ_i's and π the representation of B_n in the Hecke algebra $H(x,n)$, $\sigma_i \to g_i$. The Jones one-variable polynomial of the previous section is then a special case of the HOMFLY polynomial [16],[19]

$$V(q) = X_L(q,q) \qquad (27)$$

For instance for the Hopf link the HOMFLY polynomial is calculated to be

$$\begin{aligned} X_{\text{Hopf link}}(x,\lambda) &= \left(-\frac{1-\lambda x}{\sqrt{\lambda}(1-x)}\right)^{2-1} (\sqrt{\lambda})^2 \mathrm{tr}(g_1^2) \\ &= -\frac{1-\lambda x}{\sqrt{\lambda}(1-x)}\lambda \, \mathrm{tr}\left((x-1)g_1 + x\right) \\ &= -\frac{1-\lambda x}{\sqrt{\lambda}(1-x)}\lambda \left((x-1)(-)\frac{1-x}{1-\lambda x} + x\right) \\ &= -\frac{\sqrt{\lambda}}{1-x}\left(1 - x + x^2(1-\lambda)\right) \qquad (28) \end{aligned}$$

and the Jones polynomial $V(q) = X_L(q,q) = -q^{1/2} - q^{5/2}$ which agrees with our calculation in the previous section using the skein relations or the bracket polynomial.

4 The Fundamental group of $\mathcal{D}_n$

The fundamental group of a topological space X can be introduced by making the homotopy equivalence classes of paths that start and end at a fixed point in a space into a group. Indeed, for a point x in X, a loop at x is a path that starts and ends at x. Then the *fundamental group of X with base point x*, denoted $\pi_1(X,x)$, is defined to be the set of equivalence classes of loops at x, where the equivalence is by homotopy. Also the notions of covering spaces and fundamental groups are intimately related: coverings correspond to subgroups of the fundamental group. There is a universal covering, from which all other coverings can be constructed [20].

In [6] the universal covering space $\tilde{\mathcal{D}}_n$ of the domain $\mathcal{D}_n$ of Appell-Lauricella's function F_D was determined. There it was shown that $\tilde{\mathcal{D}}_n$ is isomorphic to $\mathcal{E}$ which is the space of the quotient of all simple, closed, rectilinear curves on Riemann's sphere by a certain equivalence. Subsequently, a presentation for $\pi_1(\mathcal{D}_n,a)$ was determined, where it was showed that $\pi_1(\mathcal{D}_n,a)$ is isomorphic to P_{n+2}/Z_{n+2}. The normal subgroup Z_{n+2} of the pure braid group denotes its centre.

We follow the notation used in [6]. Consider the set $N_n=\{0,1,\ldots,n+1\}$ where n is a non-negative integer. Being given a, C_a on the Riemann's sphere U and two different integers $i,j\in N_n$, one considers a simple curve (path): $u=u_{ij}(t),(0\le t\le 1)$ such that $u_{ij}(0)=a_i, u_{ij}(1)=a_j, u_{ij}(t)=u_{ji}(1-t)$ and that $u_{ij}(t)$ is contained in the domain $U(C_a)$ provided that $0<t<1$ and a family of functions $h_{C_a,ij,s}(t)$ of which the curve: $u=h_{C_a,ij,s}(t)$ is a lace in comparison with $u_{ij}(s)$ leaving a_i. For $I=\{i_\alpha\}\subset N_n$, one supposes always $i_\alpha<i_\beta$ if $\alpha<\beta$. For each pair $i,j\in N_n$, one denotes by A_{ij} the element of $\pi_1(\mathcal{D}_n,a)$ represented by the curve: $z_\alpha=a_\alpha(i\ne\alpha), z_i=h_{C_a,ij,1}(t)$ [7].

Being given a group G_n generated by $\{A_{ij}; i,j\in N_n, i\ne j\}$, one poses, for $I=\{i_\alpha;\alpha\in N_p\}$,

$$\begin{aligned} A_{i_0i_1\cdots i_p;i_{p+1}} &:= A_{i_0i_{p+1}}A_{i_1i_{p+1}}\cdots A_{i_pi_{p+1}}, \\ A_I = A_{i_0i_1\cdots i_{p+1}} &:= A_{i_0;i_1}A_{i_oi_1;i_2}\cdots A_{i_0i_1\cdots i_p;i_{p+1}} \end{aligned} \tag{29}$$

It is said that G_n admits the relation $\mathbf{R}_0^n$ if one has $A_{ij}=A_{ji}\forall i,j\in N_n$ and that G_n admits $\mathbf{R}_q^n(I)$ if one has, for all $J=\{j_\beta;\beta\in N_q\}$ with $q\le p$ and $J\subset I, A_J\leftrightarrow A_{j_\alpha j_\beta}$ where $\leftrightarrow$ signifies commutativity [6]. In addition, we say that G_n admits $\mathbf{R}_n^n$ if one has the relation $A_{01\cdots n+1}=1$ the unity of G_n [8].

[7] As it is remarked in [6] A_{ij} can be regarded as an element of the coloured braid group. Indeed two sets (z) and (z') of $n+2$ complex numbers among whom no two can be equal were defined to be equivalent if and only if $z_0-z_0'=\cdots=z_{n+1}-z_{n+1}'$. Defining A_{ij} by the curve with the same formulae as above in this new space one can regard A_{ij} as an element of the pure braid group. When $i<j$, using the elements $\sigma_\alpha(\alpha=0,\cdots,n)$ of the braid group B_{n+2}, one can write $A_{ij}=\sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_i$ and $A_{ji}=\sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$.

[8] Using (29) the relation $\mathbf{R}_n^n$ is equivalent to: $A_{0;1}A_{01;2}\cdots A_{01\cdots n;n+1} = A_{01}A_{02}A_{12}\cdots A_{0n+1}A_{1n+1}\cdots A_{nn+1}=1$.

Being given $I \subset N_n$ and a positive η one writes

$$\begin{aligned} S_I(\eta) \quad = \quad & \bigcap_{\alpha \in N_p} \{z = (z_0, z_1, \cdots, z_{n+1}); z \in \mathcal{D}_n, \\ & |z_{i_\alpha} - z_{i_0}| < \eta \sup \{|z_i - z_{i_0}|; i \in N_n \setminus I\}\} \end{aligned} \tag{30}$$

and $\mathring{\mathrm{S}}_I(\eta)$ is the set of the interior points.

Then using a series of lemmas the author transports the paths and homotopies of $\mathcal{D}_n$ to those of $S_I(\eta)$ and he proves that the fundamental group $\pi_1(\mathcal{D}_n, a)$ is generated by [6]:

$$\{A_{ij}; i, j \in N_n, i \neq j\}$$

and the relations among these elements are reduced to the set of relations:

$$\mathbf{R}_0^n, \mathbf{R}_1^n(N_n), \mathbf{R}_2^n(N_n), \mathbf{R}_n^n.$$

Consequently one can chose $(n+1)(n+2)/2 - 1$ elements as generators.

Subsequently the author in [6] as a corollary determines that the coloured braid group is generated by

$$\{A_{ij}; i, j \in N_n, i \neq j\}$$

and the relations among the elements are reduced to those of the set of relations

$$\mathbf{R}_0^n, \mathbf{R}_1^n(N_n), \mathbf{R}_2^n(N_n).$$

5 Conclusions

In this work using and combining results from [1], [2], [3] and [6] we have established a direction connection of the theory of General Relativity with the theory of the pure braid group. More specifically, the connection is established via the generalised multivariable hypergeometric function of Lauricella F_D through which the exact solutions of the equations of motion of test and photon particles in Kerr and Kerr-(anti) de Sitter spacetimes and of the corresponding physical quantities such as periapsis and gravitomagnetic precessions, bending of light and deflection angle were expressed [1], [2], [3] . As we discussed in the main text the topological properties of the domain of variables $\mathcal{D}_n$ are such that the fundamental group $\pi_1(\mathcal{D}_n, a)$ is isomorphic to the quotient group P_{n+2}/Z_{n+2} [6].

The domain of variables $\mathcal{D}_n$ of F_D is related in the theory of General Relativity to the first integrals of motion as well as to the cosmological constant and the Kerr (spin) parameter of the rotating black hole or rotating central star.

We also mentioned in the main body of the paper that the closure operation on pure braids lead to links.

We believe that the link established in this work between General Relativity the leading fundamental physical theory of gravity and low dimensional topology which involves the theory of coloured braids and links is very important. It

may also provide us with hints and clues about the observed dimensionality of spacetime and the topological origin of some physical quantities. On the topology side it might lead to new invariants for links through the hypergeometric representation of the pure braid symmetry group. The first light from the dawn of a new era has reached us.

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References

[1] G. V. Kraniotis, *Precise relativistic orbits in Kerr and Kerr-(anti) de Sitter spacetimes*, Class. Quantum Grav. **21** (2004) 4743-4769, [arXiv:gr-qc/0405095]

[2] G. V. Kraniotis, *Frame dragging and bending of light in Kerr and Kerr-(anti) de Sitter spacetimes*, Class. Quantum Grav.**22** (2005) 4391-4424,[arXiv:gr-qc/0507056]

[3] G. V. Kraniotis, *Periapsis and gravitomagnetic precessions of stellar orbits in Kerr and Kerr-de Sitter black hole spacetimes*,Class.Quantum Grav.**24** (2007) 1775-1808, [arXiv:gr-qc/0602056]

[4] P. Appell, *Sur les fonctions hypergéométriques de deux variables*, Journal de Mathématiques pures et appliquées de Liouville, VIII (1882) p 173-216

[5] G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, Rend. Circ. Mat. Palermo **7** (1893), 111-158

[6] T. Terada, *Quelques Propriétés Géométriques du Domaine de F_1 et le Groupe de Tresses Colorées*, Publ. RIMS, Kyoto Univ. **17** (1981), 95-111

[7] M. F. Atiyah, *The Geometry and Physics of Knots*, Cambridge Univ. Press, Cambridge 1990

[8] V. V. Prasolov and A. B. Sossinsky, *Knots, Links, Braids and 3-Manifolds, An Introduction to the New Invariants in Low-Dimensional Topology*, Translations of Mathematical Monographs **154**, AMS 1997

[9] E. Artin, *Theorie der Zöpfe*, Hamburg Abh **4**, (1925), 47-72

[10] E. Artin, *Theory of braids*, Ann. of Math. **48**, (1947), 101-126

[11] J. S. Birman, *Braids, Links and Mapping Class Groups*, Ann. Math. Stud. **82** (1974), Princeton University Press

[12] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen*, Hamburg Abh **11**, (1936) 179-186

[13] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **20**, (1923) 93-95

[14] O. Schreier, *Die Untergruppen der freien Gruppen*, Hamburg Abh **5**,(1927), 161-183

[15] B. J. Gassner, *On braid groups*, Hamburg Abh **25**, (1961) 10-22

[16] V. F. R. Jones, *A polynomial invariant for knots via Von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103-112

[17] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239-246

[18] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395-407

[19] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335-388

[20] W. Fulton, *Algebraic Topology*, Springer GTM **153**, 1997